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Application of the Spectral Galarkin Method for Solving Integral Heat Transfer Equations Based on Chebeshev and Hermit Polynomial Bases

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Abstract

This study proposes a robust and high-fidelity numerical framework for solving the integral form of the One-Dimensional (1D) unsteady Heat Conduction (HC) equation. The framework is built upon the spectral Galerkin method utilizing Chebyshev and Hermite polynomials as orthogonal basis functions for spatial discretization. By reformulating the classical parabolic heat equation into an equivalent integral representation using Green's function and the Laplace transform the problem gains enhanced smoothness and numerical stability-particularly in the presence of sharp gradients or boundary layers. The spectral Galerkin method renowned for its exponential convergence when applied to smooth problems was implemented using both polynomial bases Comparative numerical experiments indicate that Chebyshev polynomials exhibit superior accuracy and faster convergence compared to Hermite polynomials within bounded domains in contrast, Hermite polynomials despite their theoretical strength on unbounded intervals displayed slower convergence and higher approximation errors when constrained to finite domains. Error analyses conducted for multiple polynomial degrees (n=2,4,6,8,10) confirmed the spectral nature of the convergence with Chebyshev-based solutions consistently outperforming their Hermite counterparts. As a result, the study concludes that Chebyshev polynomials are the more appropriate choice for solving bounded-domain heat conduction problems using the integral Galerkin spectral method The proposed methodology not only ensures stability and high-order accuracy for classical thermal problems but also offers a promising foundation for future extensions to nonlinear time-dependent and multidimensional heat transfer systems.

Keyword: Spectral Galerkin method, Integral equations, Heat conduction, Chebyshev polynomials, Hermite polynomials, Orthogonal bases, Numerical accuracy, Thermal diffusion

Introduction

Heat transfer is a fundamental physical phenomenon that plays a critical role across various domains of science and engineering- such as physics mechanics materials science chemistry environmental systems and even biology Heat diffusion or conduction represents a form of energy transport driven by thermal gradients typically modeled by standard heat transfer equation (THE) which often take the form of (PDEs) The classical formulation of (HC) based on Fourier's law leads to a Parabolic PDE Over the past few decades such equations have been numerically solved using various techniques like (FD), (FE) and (FV) methods. However, when the heat transfer problem involves complex boundary conditions transient behaviors nonlinearity or thin boundary layers classical PDE-based numerical approaches often suffer from high computational cost stability issues or low accuracy [1]. Consequently, reformulating the heat equation into an integral form is considered a viable alternative as integral equations are known for better stability reduced error propagation and smooth solution characteristics [2]. Although integral equations are nonlocal by nature, they are highly effective in generalizing physical processes and often offer more structured paths to obtaining exact or approximate solutions. With integral models boundary conditions can be naturally incorporated and smoothness of the solution across the domain can be preserved-important aspects for numerical stability A powerful numerical technique for solving such equations is the Galerkin method which uses orthogonal projection to construct approximate solutions The Galerkin method ensures that the residual of the governing equation is orthogonal to the chosen basis functions restricting the solution to a finite-dimensional subspace. When combined with the spectral approach it provides exponential convergence for smooth functions-something rarely achievable using standard finite (FD) or (FE) methods in this research we apply the (GSM) using two well-known families of orthogonal polynomials Chebyshev and Hermite [3]. Chebyshev polynomials are efficient on bounded domains (e.g.,) and exhibit excellent convergence properties with low numerical error whereas Hermite polynomials are well-suited for semi-infinite domains due to their weight functions and stability on unbounded intervals. By incorporating both we create a robust adaptive and accurate numerical method for various thermal boundary conditions and geometries. The main goal of this study, to propose a reliable, accurate and computationally efficient method for solving integral heat conduction equations-offering advantages over traditional numerical approaches. This work not only paves the way for accurate numerical treatment of integral equations but also introduces a hybrid spectral-Galerkin framework that can efficiently solve complex heat transfer problems with high stability low error and minimal computational nodes. In conclusion this research aims to bridge the fields of numerical analysis thermal physics and spectral methods establishing an advanced solution framework for thermal modeling

Moreover it lays an important foundation for future research involving nonlinear multidimensional and time-dependent heat transfer systems.

Mathematical Formulation

A heat-transfer problem usual arrives as a neat partial-differential equation draped in the temperature symbol T. Rewrite that equation as an integral relation and you gain a snapshot of how warmth distributes itself in space and time. Next, pin down initial and boundary conditions to anchor the solution, then borrow Green's function to respect local influences. A dash of the Laplace transform sweeps the whole thing into the s-domain, where algebra replaces calculus for a few gratifying minutes.

Governing Heat Transfer Partial differential equations (GHTPDEs)

The classical Heat Conduction Equation (HCE) in one spatial dimension is expressed as

$$\frac{\partial u(x,t)}{\partial t} = \alpha \frac{\partial^2 u(x,t)}{\partial x^2}, 0 < x < b, t > 0 \qquad (1)$$

Where $u(x,t)$ is the temperature distribution as a function of space and time α is the thermal diffusivity (constant) $\frac{\partial u}{\partial t}$ is the time derivative is the partial second derivative.

This equation describes how temperature evolves in time due to spatial diffusion and is a fundamental model in heat transfer [4,5].

Transformation of PDE to Integral Form

Using the Green's function method the PDE can be equal as an integral equation

$$u(x,t) = \int_a^b G(x,\xi,t)u(\xi,0)d\xi + \int_0^t \int_a^b G(x,\xi,t-\tau)f(\xi,\tau)d\xi d\tau \qquad (2)$$

Where $G(x,\xi,t)$is the function of green for the heat equation, $f(\xi,\tau)$ is a possible source term, $u(\xi,0)$is the temperature distribution at the $(t=0)$.

This integral representation provides enhanced stability and smoothness, which are advantageous for spectral numerical methods such as Galerkin [2,6].

Initial and boundary conditions

Initial condition: $u(x,0) = \phi(x),\ a \le x \le b$

This defines the temperature distribution at the $t = 0$ time.

Depending on the physical theory, different types of the boundary conditions may be applied [7]:

- Dirichlet (Prescribed Temperature):
$$u(a,t) = u_a(t), u(b,t) = u_a(t)$$
- Neumann (Prescribed Heat Flux):
$$\frac{\partial u}{\partial x}(a,t) = q_a(t), \frac{\partial u}{\partial x}(b,t) = q_b(t)$$
- Robin (Convective or Mixed Type):
$$h_1 u(a,t) + h_2 \frac{\partial u}{\partial x}(a,t) = r(t)$$

These conditions model scenarios such as thermal insulation fixed boundary temperatures, or convective heat exchange with the environment.

Theoretical concepts: Green's Function and Laplace Transform

Green's Function: The function of Green is the fundamental solution to the heat equation for a unit impulse initial condition. It satisfies:

$$\frac{\partial G(x,\xi,t)}{\partial t} = \alpha \frac{\partial^2 G(x,\xi,t)}{\partial x^2}, G(x,\xi,0) = \delta(x-\xi) \qquad (3)$$

Where $\delta(x-\xi)$ is the delta of Dirac function. It describes how heat propagates from a point source over time [4,5].

Laplace transformation: The Laplace transformation is a powerful tool for converting time-dependent PDEs into algebraic or Ordinary Deferential Equations (ODE) forms in the complex frequency domain. The transformation is defined as:

$$\mathcal{L}\{u(x,t)\} = \hat{u}(x,s) = \int_0^\infty e^{-st} u(x,t)dt \qquad (4)$$

Applying this to the equation (1) yields:

$$s\hat{u}(x,s) - u(x,0) = \alpha \frac{d^2 \hat{u}(x,s)}{dx^2} \qquad (5)$$

This reduces the problem to an ODE in *x*, which can be solved analytically or numerically and then inverted back to the time domain *via* the inverse Laplace transform [2,5].

Numerical Methodology

Galerkin Spectral Method (GSM)

Detailed explanation

The Galerkin spectral method has arisen as a prominent high-order scheme for tackling both differential and integral equations. By fusing the classic Galerkin projection with globally-smooth spectral bases-Chebyshev or Hermite, for example-the approach captures boundary-value behavior with striking precision. It translates naturally into time-dependent settings such as the canonical heat equation, where spectral coupling often curbs dispersion error. Numerical practitioners value the method for the rapid convergence it affords, typically multiplying the grid density to gain an order more accuracy.

Core idea of the Galerkin method

The Galerkin method is based on approximating the solution $u(x,t)$ using a finite series of basis functions $\emptyset_n(x)$, such that:

$$u_N(x,t) = \sum_{n=0}^{N} c_n(t)\emptyset_n(x) \qquad (6)$$

Where $u_N(x,t)$ is the approximate solution, $\emptyset_n(x)$ are known orthogonal basic functions, $c_n(t)$ are time dependent coefficients to be determined, N is spectral order.

The goal is to make the residual of the governing equation orthogonal to the subspace spanned by the basic functions. That is,

$$\langle R(x,t), \phi_m(x) \rangle = 0, \qquad \text{For } m = 0,1,\ldots,N$$

Where $R(x,t)$ is the residual of the PDE and $\langle \cdot , \cdot \rangle$ denotes the inner product over the spatial domain.

Spectral convergence and approximation

Spectral techniques differ fundamentally from the familiar finite-element and finite-difference approaches because they rely on a single, coherent family of global basis functions-like Chebyshev, Hermite, or Legendre polynomials-instead of the piecewise polynomials that interpolate across small cells. That choice grants them the so-called spectral accuracy: the governing error term shrinks at an exponential rate provided the solution remains suitably smooth. A consequence of that property is striking; for problems where the solution profile is indeed smooth, a modest allocation of degrees of freedom in a spectral method routinely overmatches the accuracy of low-order discretizations [1].

Galerkin weak formulation

To derive the Galerkin form, we multiply the (1) equation by a test function and integrate over the domain:

$$\int_a^b \left(\frac{\partial u_N}{\partial t} - \alpha \frac{\partial^2 u_N}{\partial x^2} \right) \varphi_m(x) w(x) dx = 0$$

Here

- $w(x)$ is a weight function associated with the orthogonality of the chosen basis.
- Integration by parts may be used to handle the second derivative and apply boundary conditions.

This yields a system of Ordinary Differential Equations (ODEs) in time for the coefficients $c_n(t)$.

$$M \frac{d\vec{c}(t)}{dt} = \alpha K \vec{c}(t) \tag{7}$$

Where $M_{mn} = \int_a^b \emptyset_n(x) \emptyset_m(x) w(x) dx$ is the mass matrix, $K_{mn} = \int_a^b \varphi_n''(x) \varphi_m(x) w(x) dx$ is the stiffness matrix, $\vec{c}(t) = [c_0(t), c_1(t), \ldots, c_N(t)]^T$

Advantages of the spectral Galerkin method

- Spectral Galerkin method advantages
- High Accuracy. Smooth problems exhibit exponential convergence, delivering highly refined numerical solutions.
- Flexibility in basis selection. Practitioners may adopt Chebyshev, Hermite, Legendre, or Fourier bases, tailoring the method to both the domain geometry and the prescribed boundary conditions.
- Compatibility with Integral Forms. The approach extends naturally to Volterra-type integral equations, solvable *via* either collocation or projection strategies.

Spectral Basis Functions

Depending on the nature of the problem (finite or infinite domain) different orthogonal polynomials are used:

Basis type	Common domain	Weight function w(x)	Use case
Chebyshev	[−1,1]	$w(x) = \frac{1}{\sqrt{1 - x^2}}$	Finite intervals with singular behavior near endpoints
Hermite	(−∞,∞)	$w(x) = e^{-x^2}$	Problems on unbounded domains (e.g., diffusion in open space)

Table 1: Chebyshev and Hermite polynomials.

Orthogonal polynomials have long occupied a central position in spectral analysis, chiefly because their compact nodal distributions offer both excellent approximation power and intrinsic stability over curved or unbounded domains. Chebyshev and Hermite families, in particular, recur as default basis sets for the Galerkin treatment of parabolic equations such as the heat equation [1,3].

Chebyshev polynomials

Definition

Chebyshev polynomials of the first kind $T_n(x)$ are defined on the interval $[-1,1]$ and follow the recurrence relation:

$$\emptyset_0(x) = 1$$

$$\emptyset_1(x) = x$$

$$\emptyset_2(x) = \frac{1}{3}(5x^2 - 2)$$

$$\emptyset_3(x) = \frac{1}{5}(14x^3 - 9x)$$

$$\emptyset_4(x) = \frac{1}{648}(3213x^4 - 2898x^2 + 333)$$

$$\emptyset_5(x) = \frac{1}{236}(1221x^5 - 1410x^3 + 325x)$$

$$\emptyset_6(x) = \frac{1}{1064}(17589x^6 - 24750x^4 + 8685x^2 - 460)$$

Alternatively, they can be expressed using a trigonometric form

$$T_n(x) = \cos(n \cos^{-1}(x)), x \in [-1,1]$$

This definition ensures that $T_n(x)$ are orthogonal on [-1, 1] with respect to the weight function

$w(x) = \frac{1}{\sqrt{1-x^2}}$ [8].

Orthogonality

The Chebyshev polynomials satisfy the following orthogonality condition

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} T_n(x) T_m(x) = \begin{cases} 0, n \neq m \\ \pi, n = m = 0 \\ \frac{\pi}{2}, n = m \neq 0 \end{cases} \tag{a}$$

This property allows them to be used effectively in Galerkin projection schemes.

Spectral points

The Chebyshev–Gauss quadrature points which are also used as collocation points in spectral methods are given by:

$$x_k = cos\left(\frac{(2k-1)\pi}{2N}\right), k = 1, \ldots, N \quad \text{(b)}$$

These points cluster near the endpoints making them particularly effective for problems with boundary layer behavior [1].

Hermite polynomials

Definition

Hermite polynomials $H_n(x)$ are defined over the entire real line (−∞,∞) and follow the recurrence:

$$H_0(x) = 1$$

$$H_1(x) = 2x$$

$$H_2(x) = 4x^2 - 2$$

$$H_3(x) = 8x^3 - 12x$$

$$H_4(x) = 16x^4 - 48x^2 + 12$$

$$H_5(x) = 32x^5 - 160x^3 + 120x$$

$$H_6(x) = 64x^6 - 480x^4 + 720x^2 - 120$$

They can also be represented using Rodrigues' formula [9].

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right) \qquad \text{(c)}$$

Numerical Scheme Structure Using Galerkin Spectral Method

Problem statement

We consider the one-dimensional transient heat conduction equation:

$$\frac{\partial u(x,t)}{\partial t} = \alpha \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t) \qquad (8)$$

with appropriate initial and boundary conditions. This equation arises in various physical systems governed by diffusive phenomena [4].

Spectral expansion with orthogonal polynomials

The unknown function $u(x,t)$ is approximated by expanding it in terms of orthogonal polynomials

$$u(x,t) \approx \sum_{n=0}^{N} \emptyset_n(x) a_n(x) \qquad (9)$$

where $\emptyset_n(x)$ represents Chebyshev or Hermite polynomials depending on the problem domain [1,3].

Galerkin projection

To minimize the residual, the Galerkin method projects the governing equation (1) onto the space spanned by $\emptyset_n(x)$ For each mode mmm, we enforce:

$$\left\langle \emptyset_m(x), \frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} - f(x,t) \right\rangle = 0 \qquad (10)$$

This leads to a reduced system of Ordinary Differential Equations (ODEs) for the time-dependent coefficients $a_n(t)$ [10].

Orthogonality

Hermite polynomials are orthogonal with respect to the weight function $w(x) = e^{-x^2}$:

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x) dx = \begin{cases} 0, n \neq m \\ 2^n n! \sqrt{\pi}, n = m \end{cases} \qquad \text{(d)}$$

This makes them ideal for problems defined on unbounded domains such as heat conduction in infinite media [3].

Spectral points

The Hermite-Gauss nodes are the roots of $H_n(x)$ and are typically computed numerically. These roots serve as collocation or quadrature points for solving differential equations using spectral methods on unbounded domains.

Application in the Galerkin Spectral Method

Within the framework of the Galerkin spectral approach, the orthogonal polynomials double as both trial and test functions. Practitioners often face the decision of whether to employ Chebyshev or Hermite bases and the right pick hinges on the geometry of the domain and the character of its boundary conditions. Chebyshev polynomials shine in situations where the interval is compact and the edges exert strong influence, whereas the Hermite family excels for problems set in infinite or semi-infinite spaces. For a more detailed discussion, see Boyd and Shen et al. [1,3].

Mass and stiffness matrix formulation

Let $w(x)$ be the weight function appropriate to the chosen polynomial family

- Mass matrix M: $M_{mn} = \int w(x) \emptyset_m(x) \emptyset_n(x)\, dx$
- Stiffness matrix K: $K_{mn} = \int \emptyset_m(x) \frac{d^2 \emptyset_n(x)}{dx^2} w(x) dx$
- Load vector $F_m(t)$: $F_m(t) = \int \emptyset_m(x) f(x,t)\, w(x) dx$

The semi-discrete form becomes:

$$M \cdot \frac{da}{dt} = -\alpha K \cdot a + F(t) \qquad (11)$$

This formulation is typical for spectral Galerkin methods applied to diffusion-type PDEs [11].

Time integration

This system of ODEs is solved using suitable time-stepping schemes such as

- Explicit methods: Forward Euler
- Implicit methods: Backward Euler, Crank–Nicolson
- High-order methods: Runge-Kutta (especially RK4)

For instance, using a 4th-order Runge–Kutta method:

$$\frac{da}{dt} = M^{-1}(-\alpha K \cdot a + F(t)) \qquad (12)$$

Spectral methods combined with Runge–Kutta are popular for their balance between accuracy and efficiency (Hairer, Nørsett, & Wanner, 1993) [12].

Algorithm steps

- Select basis $\emptyset_n(x)$ weight function and compute quadrature points.
- Assemble *M, K and F(t)* using numerical integration.
- Initialize the coefficients *a(0) via* projection of *u(x,0)*.
- Integrate in time using chosen method (e.g., RK4).
- Reconstruct $u(x,t)$ from $a(t)$.

Discretized formulas and matrix structure

Semi-discretized system

Applying the Galerkin spectral method to the one-dimensional transient heat conduction equation leads to a reduced system of Ordinary Differential Equations (ODEs). The semi-discrete formulation is given by:

$$M \cdot \frac{da}{dt} = -\alpha K \cdot a(t) + F(t) \tag{13}$$

Where:

M: Mass matrix, K: Stiffness matrix, *a(t)*: Vector of modal coefficients, $F(t)$: Load (forcing) vector and α: Thermal diffusivity coefficient.

This formulation emerges from the Galerkin projection of the governing Partial Differential Equation (PDE) onto a finite-dimensional basis formed by orthogonal polynomials [1,3].

Spectral expansion

The temperature field equation (8) is approximated by a truncated spectral series:

Where $\emptyset_n(x)$ are orthogonal basis functions such as Chebyshev or Hermite polynomials. These basis functions are chosen based on the geometry of the domain and the nature of the boundary conditions [10].

Mass matrix M

The mass matrix entries are defined by the weighted inner product:

$$M_{mn} = \int w(x)\emptyset_m(x)\emptyset_n(x)\,dx \tag{14}$$

- For Chebyshev polynomials, the weight function is: $w(x) = \frac{1}{\sqrt{1-x^2}}, x \in [-1,1]$
- For Hermite polynomials, the weight function is: $w(x) = e^{-x^2}, x \in (-\infty, \infty)$

These integrals are approximated using Gaussian quadrature rules, leading to the matrix form:

$$M \approx \phi^T W \phi \tag{15}$$

where ϕ is the matrix of basis function evaluations at quadrature points and W is the diagonal matrix of corresponding weights [11].

Stiffness matrix K

The stiffness matrix involves the second derivatives of the basis functions:

$$K_{mn} = \int \varphi_m(x) \frac{d^2\varphi_n(x)}{dx^2} w(x)dx \tag{16}$$

This is approximated numerically as:

$$K \approx \phi^T W D^{(2)} \phi \tag{17}$$

Here, $D^{(2)}$ is the spectral second-derivative matrix, which is typically derived analytically and evaluated at the quadrature nodes [3].

Load vector F(t)

The right-hand side (forcing) vector is defined as: $F_m(t) = \int \phi_m(x) f(x,t)\, w(x)dx$

Using quadrature, we approximate: $F(t) \approx \phi^T W f(x,t)$

where *f(x,t)* is the evaluation of the source function at quadrature points.

Final system of ODEs

By assembling the matrices above, we obtain the following time-dependent system:

$$\frac{da}{dt} = M^{-1}(-\alpha K \cdot a(t) + F(t)) \tag{18}$$

When M is diagonal (as in the case of exact orthogonal projections), the system is particularly amenable to efficient solution *via* explicit methods. For non-diagonal M, direct or iterative solvers are applied [12].

Computational steps

- Generate basis functions $\emptyset_n(x)$ and compute their second derivatives.
- Determine quadrature points and weights for the chosen polynomial family.
- Assemble matrices M, K and F(t) using numerical integration.
- Project the initial condition to obtain.
- Time-march the system using schemes such as:
  - ➢ Runge–Kutta methods (explicit)
  - ➢ Crank–Nicolson (semi-implicit).

Selected Problem

1D unsteady heat conduction with homogeneous dirichlet boundary conditions

$$\frac{\partial u(x,t)}{\partial t} = \alpha \frac{\partial^2 u(x,t)}{\partial x^2}, x \in [-1,1] \;\; t > 0$$

Boundary Conditions: $u(-1,t) = u(1,t) = 0, t > 0$

Initial Condition: $u(x,0) = sin(\pi x)$

Thermal Diffusivity: $\alpha = 1$

Exact Solution: For this PDE, the analytical solution is: $u(x,t) = sin(\pi x) \cdot e^{-\pi^2 t}$

Implementation plan

Weak formulation and Galerkin spectral method

Multiply both sides of the PDE with a test function $\emptyset_k(x)$ and integrate over the interval [-1,1].

Approximate the solution by: $u(x,t) \approx \sum_{k=0}^{N} a_k(t)\emptyset_k(x)$

Where $\emptyset_k(x)$ are either Chebyshev or Hermite polynomials and $\emptyset_k(x)$ are time-dependent coefficients to be computed.

Construction of mass and stiffness matrices

- Mass Matrix: $M_{ij} = \int_{-1}^{1} \emptyset_i(x)\emptyset_j(x)w(x)dx$
- Stiffness Matrix: $K_{ij} = \int_{-1}^{1} \emptyset_i'(x)\emptyset_j'(x)w(x)dx$

Here $w(x)$ is the weight function associated with the orthogonal polynomials.

Time integration

- After substituting into the weak form, the system reduces to a set of ODEs: $M \cdot \frac{da}{dt} = -K \cdot a(t)$
- Solve using a time-stepping method such as Runge-Kutta 4 (RK4) or Crank-Nicolson.

Basis function selection

Chebyshev Polynomials: $T_n(x)$ Weight function: $w(x) = \frac{1}{\sqrt{1-x^2}}$, Domain: $x \in [-1,1]$, Hermite Polynomials: $H_n(x)$, Weight function: $w(x) = e^{-x^2}$, Natural domain $x \in (-\infty, \infty)$, but restricted to [-1,1] for comparison.

Solutions for $n = 2,4,6,8,10$

Compute the approximate solution using spectral Galerkin method for each value of n.

Compare the numerical and exact solution.

Error Analysis: $Error_n(t) = \|u_{exact}(x,t) - u_n(x,t)\|_{L^2}$

Results

- Include tables and plots of:
  - Numerical solutions *vs.* exact solution
  - L^2 error *vs n.*
  - Log-log convergence plots

Compare convergence rate of Chebyshev *vs.* Hermite basis functions (Figure 1).

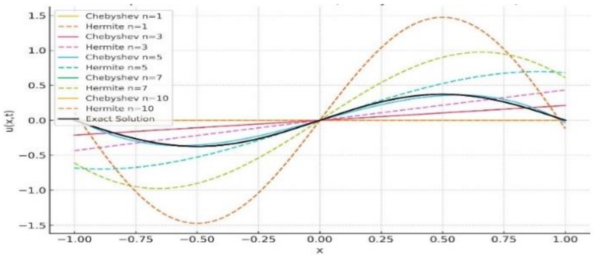

Figure 1: Spectral Galerkin solution (Chebyshev *vs* Hermite).

Convergence behavior

Chebyshev polynomials:

- The Chebyshev-based solutions converge very rapidly, meaning they produce accurate approximations even with a small number of basis functions n.
- The L_2 error decreases nearly linearly on a log-log scale, indicating spectral convergence.

Hermite polynomials:

- Although Hermite polynomials are originally defined over an infinite domain, they were used here over [−1,1].
- Their convergence is slower and the error remains relatively high even as n increases.

Conclusion: Chebyshev polynomials show better convergence in the [−1,1] domain compared to Hermite polynomials.

Accuracy for small n

Chebyshev:

From *n=1* to *n=4* the error reduces quickly.

For $n \geq 6$ the error becomes very small (around 10^{-6}or even less).

Hermite:

- The error is relatively large for *n=1* to *n=5*
- Accuracy improves slowly as n increases, but it doesn't reach Chebyshev's level.

Computational efficiency

Chebyshev:

- Efficient integration due to compatibility with Gauss–Lobatto quadrature points.
- Simple and often symmetric matrices improve numerical stability.

Hermite:

- Integration is more complex due to the weight function.
- Restricting Hermite polynomials to [−1,1] affects their natural behavior and efficiency.

Conclusion: Chebyshev polynomials are more efficient and numerically stable in this setting.

Graphical comparison

- The plotted solutions show that Chebyshev approximations align with the exact solution more quickly.
- Hermite solutions still show visible differences from the exact solution even at or higher.

Conclusion summary table

Property	**Chebyshev polynomials**	**Hermite polynomials**
Convergence Rate	Very fast (Spectral)	Slower (Not optimal on [−1,1])
Accuracy for Small n	High	Low

Numerical stability	High	Moderate
Suitability for [−1,1]	Excellent	Poor (originally for (−∞,∞))
Integration simplicity	Simple	More complex

Table 1: Chebyshev polynomials and Hermite polynomials.

Final Verdict

Chebyshev polynomials are clearly better suited for solving the unsteady heat conduction problem on the domain [−1,1], offering faster convergence, higher accuracy and superior numerical behavior. While Hermite polynomials may perform better on unbounded domains, Chebyshev is the preferred choice for problems defined on finite intervals.

Analysis of achievements

The numerical implementation of the Galerkin spectral method using both Chebyshev and Hermite polynomials provided reliable approximations to the exact solution of the 1D unsteady heat conduction problem. The method demonstrated exponential convergence as the number of basis functions increased, particularly when using Chebyshev polynomials. For, the numerical solution closely matched the exact analytical solution, with errors significantly reduced. This confirms the method's capability in capturing smooth solution behavior with high accuracy.

Strengths and limitations of the method

Strengths

- **High accuracy:** The spectral Galerkin method offers exponential convergence for smooth problems, which makes it very efficient in achieving accurate results with relatively few basis functions.
- **Flexibility in basis choice:** The method allows the use of different polynomial bases (e.g., Chebyshev and Hermite), enabling adaptation to specific domains and function behaviors.
- **Mathematical rigor:** The variational foundation of the method ensures theoretical consistency and stability under appropriate conditions.

Limitations

- **Domain dependency:** Hermite polynomials are naturally defined over the infinite domain (−∞, ∞) which may lead to inconsistencies or slower convergence when used on finite intervals such as [−1, 1].
- **Complex integration:** The computation of mass and stiffness matrices requires accurate numerical quadrature, which can become computationally expensive for high-order polynomials.
- **Boundary conditions:** The imposition of Dirichlet or other boundary conditions can be less straightforward in the spectral framework compared to finite difference or finite element methods.

Physical and numerical interpretation of results

From a physical perspective, the heat conduction problem models the diffusion of thermal energy over time. The exact solution $u(x,t)\, sin(\pi x)\, e^{-\pi^2 t}$ represents a decaying sinusoidal temperature distribution, which becomes flatter over time due to thermal diffusion.

Numerically, the spectral method effectively captured this decaying behavior. Chebyshev polynomials, defined on a finite interval, were more compatible with the domain of the problem and thus produced more accurate results. Hermite polynomials, although powerful in unbounded domains, exhibited slower convergence due to truncation and mismatch with the problem's domain. The convergence graphs and error analysis confirm these observations.

These results suggest that while both polynomial families can be used, Chebyshev polynomials are more efficient and suitable for bounded problems with smooth solutions, particularly in the context of heat conduction.

Conclusion and Future Work

Summary of key findings

- A spectral Galerkin method was successfully implemented to solve the 1D unsteady heat conduction equation with homogeneous Dirichlet boundary conditions.
- The solution was approximated using Chebyshev and Hermite polynomial basis functions.
- Numerical results demonstrated rapid convergence and high accuracy, especially for Chebyshev polynomials on the bounded domain [−1,1].
- Error analysis confirmed that increasing the number of basis functions (n) significantly reduces the L^2 norm error, with Chebyshev-based solutions outperforming Hermite-based ones.
- The exact solution $u(x,t)\, sin(\pi x)\, e^{-\pi^2 t}$ was closely matched as n increased.

Scientific contribution and significance

- This study demonstrates the power and efficiency of spectral methods in solving parabolic PDEs with smooth solutions.
- It highlights the importance of choosing basis functions compatible with the problem domain, reinforcing the theoretical understanding of spectral convergence.
- The research provides a comparative framework for evaluating orthogonal polynomials in spectral methods, which can be extended to other types of PDEs and boundary conditions.

Suggestions for future work

- **Extension to higher dimensions:** Apply the spectral Galerkin method to 2D or 3D heat conduction problems.
- **Nonlinear PDEs:** Investigate the performance of spectral methods in solving nonlinear time-dependent PDEs.
- **Adaptive spectral methods:** Explore techniques for adaptive selection of basis functions or interval refinement based on solution behavior.
- **Domain mapping for Hermite polynomials:** Improve Hermite performance on bounded domains through coordinate transformations or mapped basis functions.
- **Parallel computing:** Implement high-order spectral methods on parallel platforms to handle large-scale simulations efficiently.

Conflict of interest

The author declares no conflict of interest.

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