Interpretation of Gardner's equation

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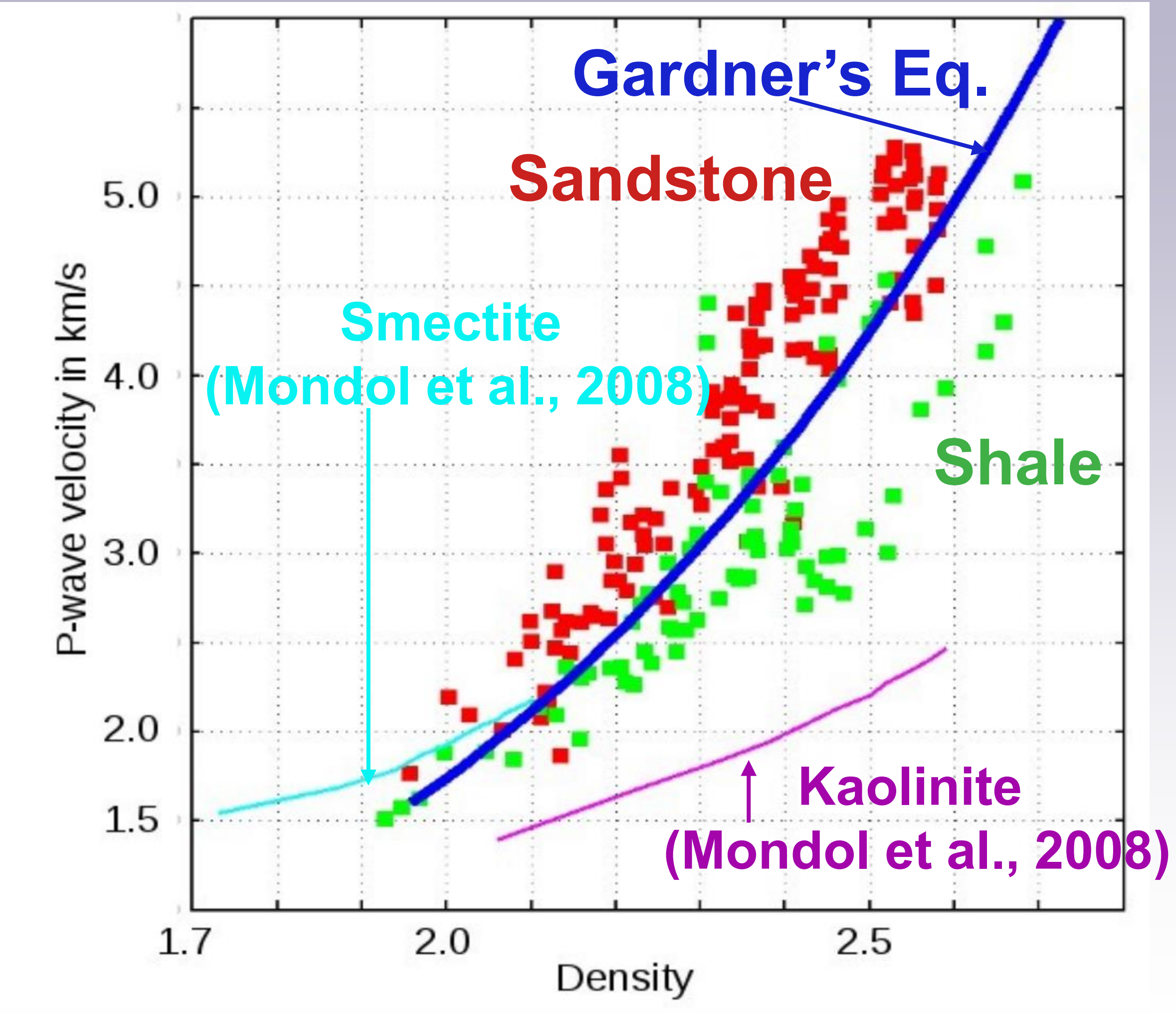
Defined Quantities

- V_p P-wave velocity
- V_b Brine P-wave velocity (~1500 m/s)
- V_s S-wave velocity
- ρ Density
- ϕ Porosity
- K Bulk Modulus
- μ Shear Modulus

Subscripts

- w wet – fluid filled pores
- d dry - no fluid in pores
- m mineral
- f fluid
- c critical

Gardner's Equation

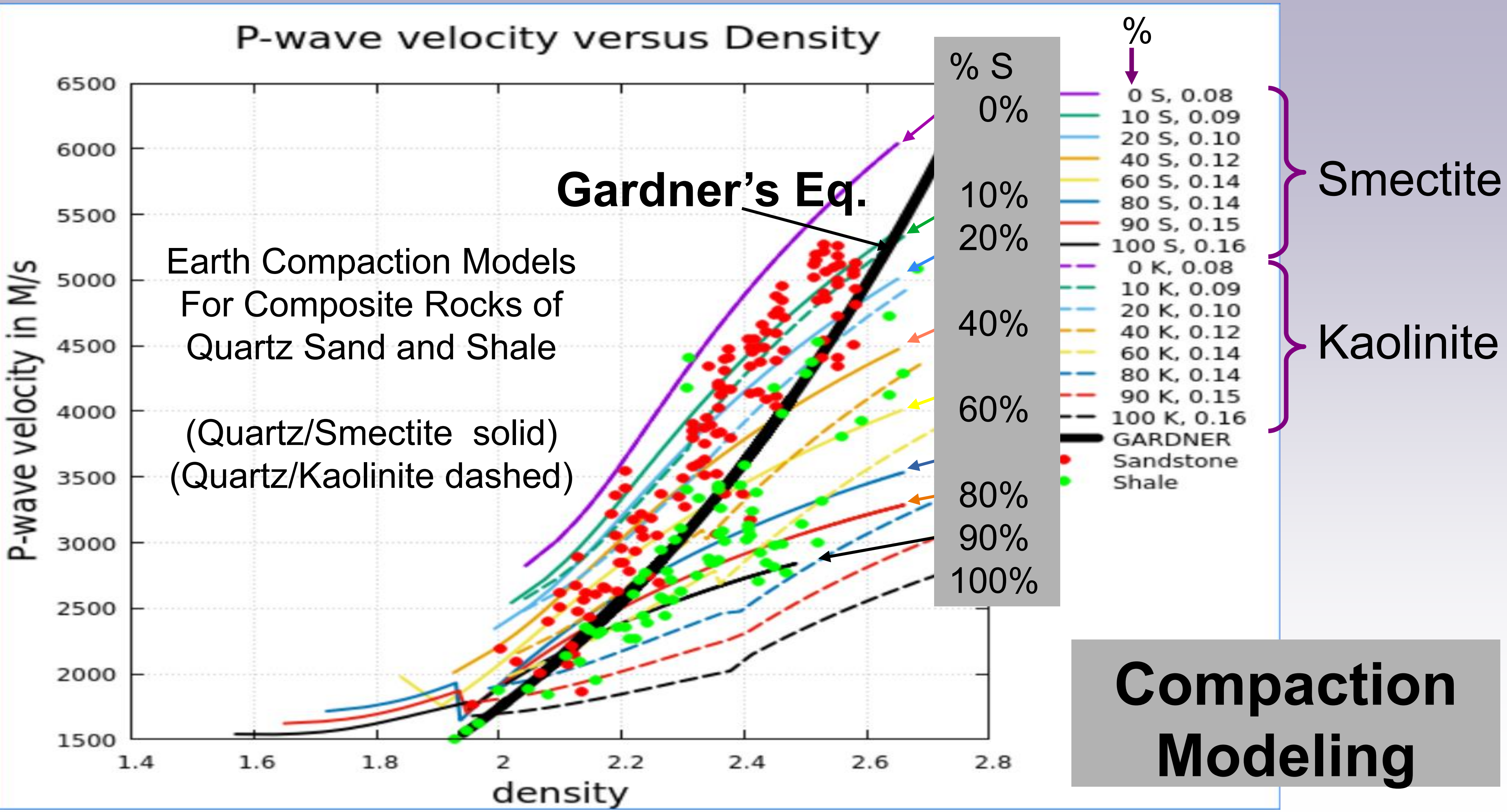


Gardner's Equation (Gardner et al., 1974, equation 7):

$$\rho = \rho_o V^{\frac{1}{4}}, \qquad \rho_o = (1.741\, Km/s)^{-1/4}$$

Density in g/cc, Sandstone and Shale data points from Castagna et al. (1993, Fig 9).

The fit is not to the points shown but instead It is a fit to "… the more prevalent sedimentary rock types through a wide range of basins, geologic ages, and depths (to 25,000 ft) …."



Compaction Modeling

Used To Analyze Gardner's Eq.

- Elastic Wave Propagation definitions of V_p and V_s
- Nur's Critical Porosity model
- Gassmann's Equations
- Aplin's Equation relating porosity to effective stress

- Density vs velocity from earth compaction modeling by *combining Nur with Aplin*
- A different density vs velocity relation that results from *combining Nur with Gassman*

Combining Gassmann’sEquations with Nur’s Critical Porosity Model Leads To These Equations

Mavko and Mukerji (1998), Krief (1990), Higginbotham et al. (2011, 2012)

$$\frac{V_p^2}{a^2} - \frac{V_s^2}{b^2} = 1$$

Higginbotham et al. (2012)

$$\rho = \frac{C_\rho}{1 - \left(\frac{sV_p}{V_b}\right)^2}$$

$$V_b = 1500\, m/s$$

a, b, s, C_ρ all constant for a given mineral and pore fluid.

Combining Gassmann'sEquations with Nur's Critical Porosity Model Leads To These Equations

Mavko and Mukerji (1998), Krief (1990), Higginbotham et al. (2011, 2012)

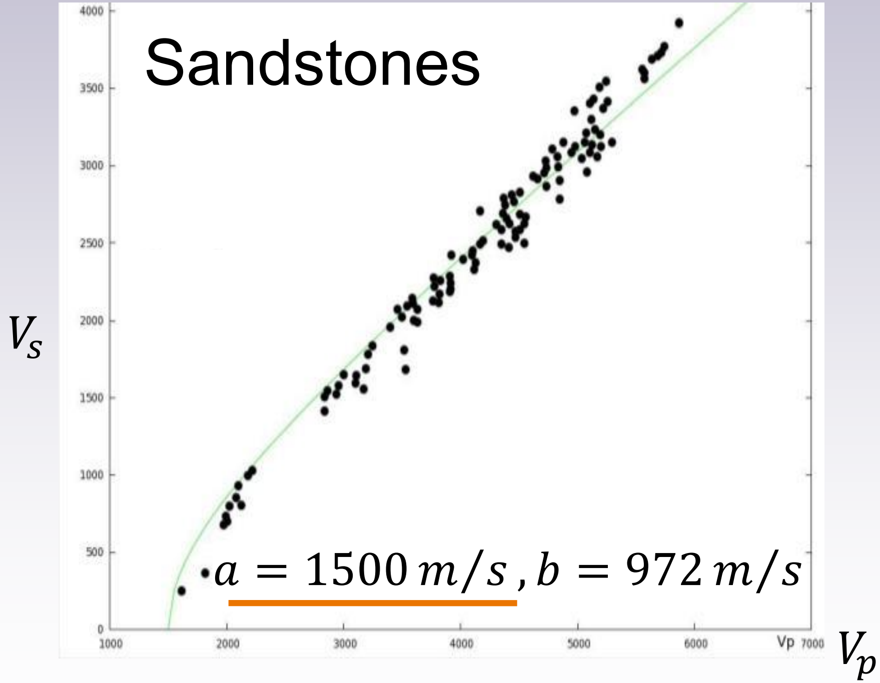
$$\frac{V_p^2}{a^2} - \frac{V_s^2}{b^2} = 1$$

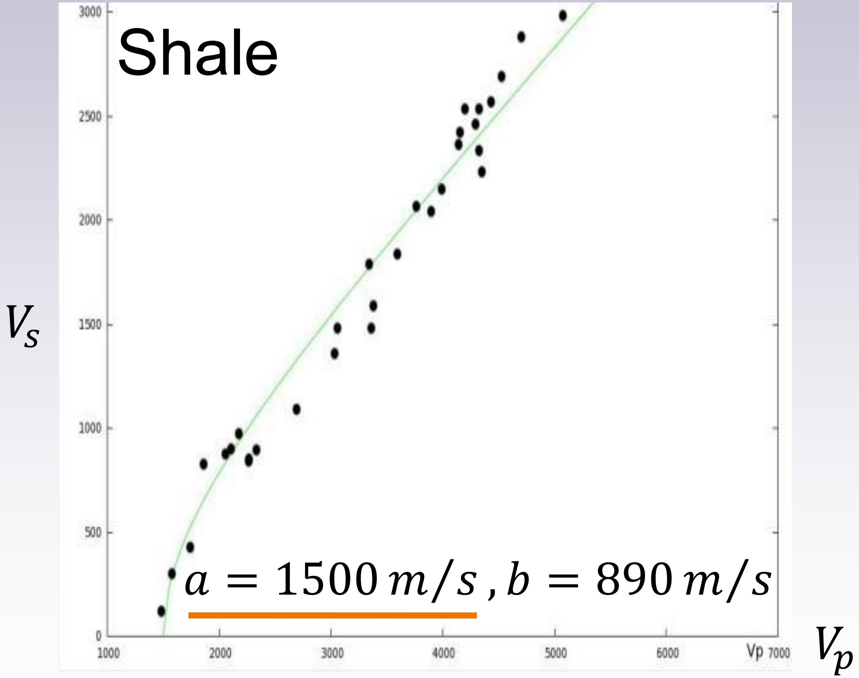
Brief attention for this equation

EVIDENCE

$$\frac{V_p^2}{a^2} - \frac{V_s^2}{b^2} = 1$$

These plots are from Higginbotham (2011), data points from Castagna (1993, p 138-139).





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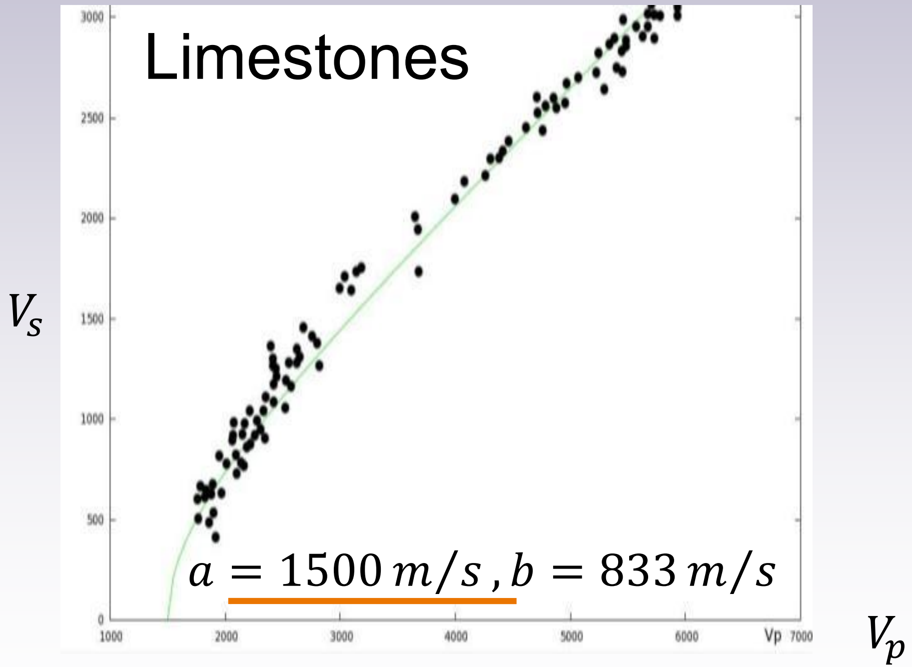
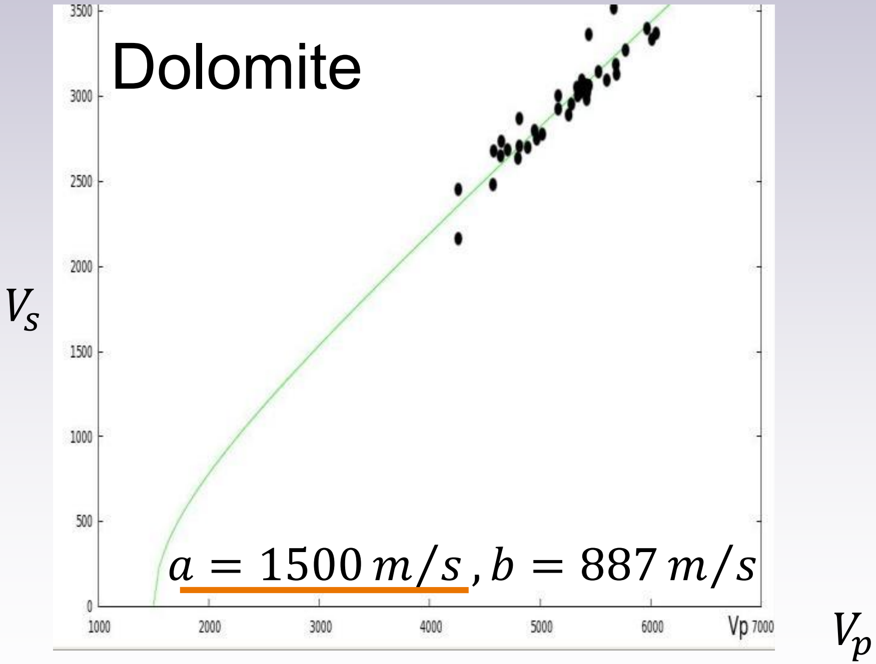
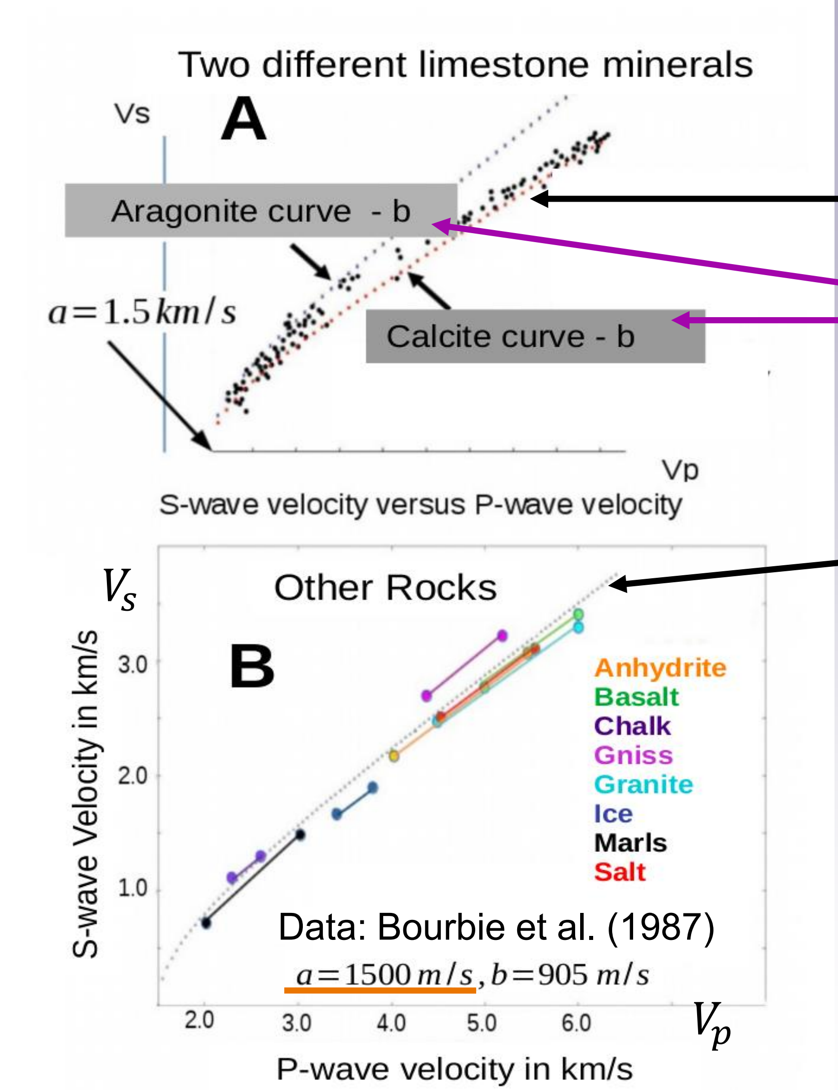






EVIDENCE

continued

Scatter is probably due to impurities.

$$\frac{1}{b^2} = \frac{1}{a^2}\left(\frac{4}{3} + \frac{K_m}{\mu_m}\right) - \frac{\rho_m}{\mu_m}$$

Note: Faint dashed curve is a fit for *b =905 m/s* to all the data shown in the previous slide – not a fit to the data points plotted in Figure **B**.

Appears to work for a variety of rocks even with a fixed at 1500 m/s.

Gassmann's Eq's are not supposed to work for these rocks ???

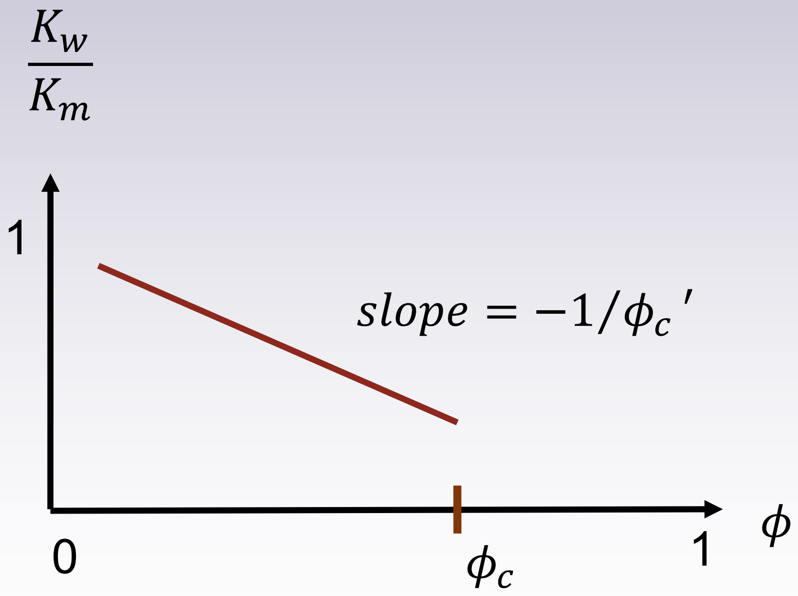
Quote from Mavko et al. (2009, pg 273)

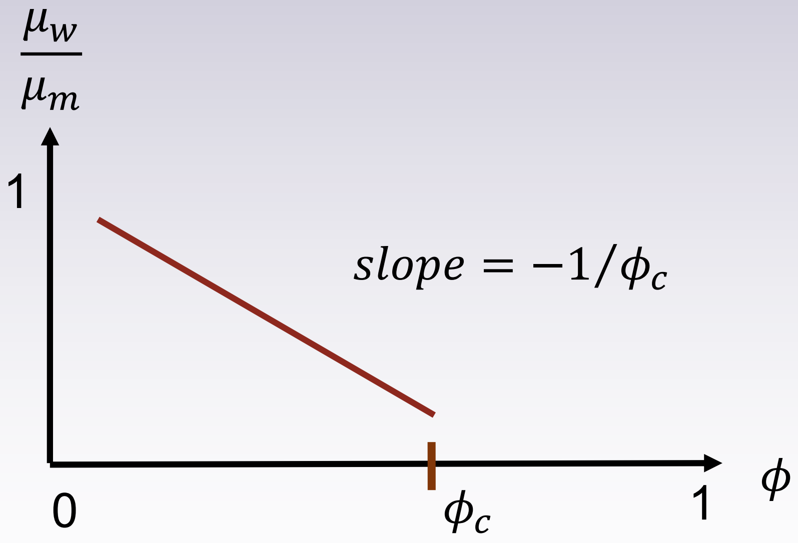
Gassmann's equation … is valid only at sufficiently ***low frequencies*** *… such that there is sufficient time for the pore fluid to flow and eliminate wave-induced pore pressure gradients …*

Combining Gassmann'sEquations with Nur's Critical Porosity Model

$$K_w/K_m = \left(1 + \left(\frac{-1}{\phi_c{}'}\right)\phi\right)$$

$$\mu_w/\mu_m = \left(1 + \left(\frac{-1}{\phi_c}\right)\phi\right)$$





Gassman-Nur also provides

$$\phi_c' = \phi_c + \frac{K_f}{K_m - K_f}$$

Higginbotham et al. (2012)

$$K_f = \frac{\phi_c' - \phi_c}{1 + \phi_c' - \phi_c} K_m$$

The *MODIFIED* Gassman-Nur Model replaces K_f with this expression.

THE FLUID BULK MODULUS VANISHES FROM THE EQUATIONS

It is reasonable to conclude that the fluid bulk modulus K_f is the source of error in Gassmann's Eqs for low permeable rocks and high frequencies.

Eliminate K_f

$$K_f \rightarrow \frac{\phi_c{}' - \phi_c}{1 + \phi_c{}' - \phi_c} K_m$$

Removing Dependence on Fluid Bulk Modulus

$$K_f \rightarrow \frac{{\phi_c}' - \phi_c}{1 + {\phi_c}' - \phi_c} K_m$$

Modified Gassman-Nur Model

No dependence on K_f

$$a = a(\phi_c, {\phi_c}', K_f, K_m, \rho_f, \rho_m) \rightarrow a = a(\phi_c, {\phi_c}', K_m, \rho_f, \rho_m)$$

$$b = b(\phi_c, {\phi_c}', K_f, K_m, \rho_f, \rho_m, \mu_m) \rightarrow b = b(\phi_c, {\phi_c}', K_m, \rho_f, \rho_m, \mu_m)$$

$$C_\rho = C_\rho(\phi_c, {\phi_c}', K_f, K_m, \rho_f, \rho_m, \mu_m) \rightarrow C_\rho = C_\rho(\phi_c, {\phi_c}', K_m, \rho_f, \rho_m, \mu_m)$$

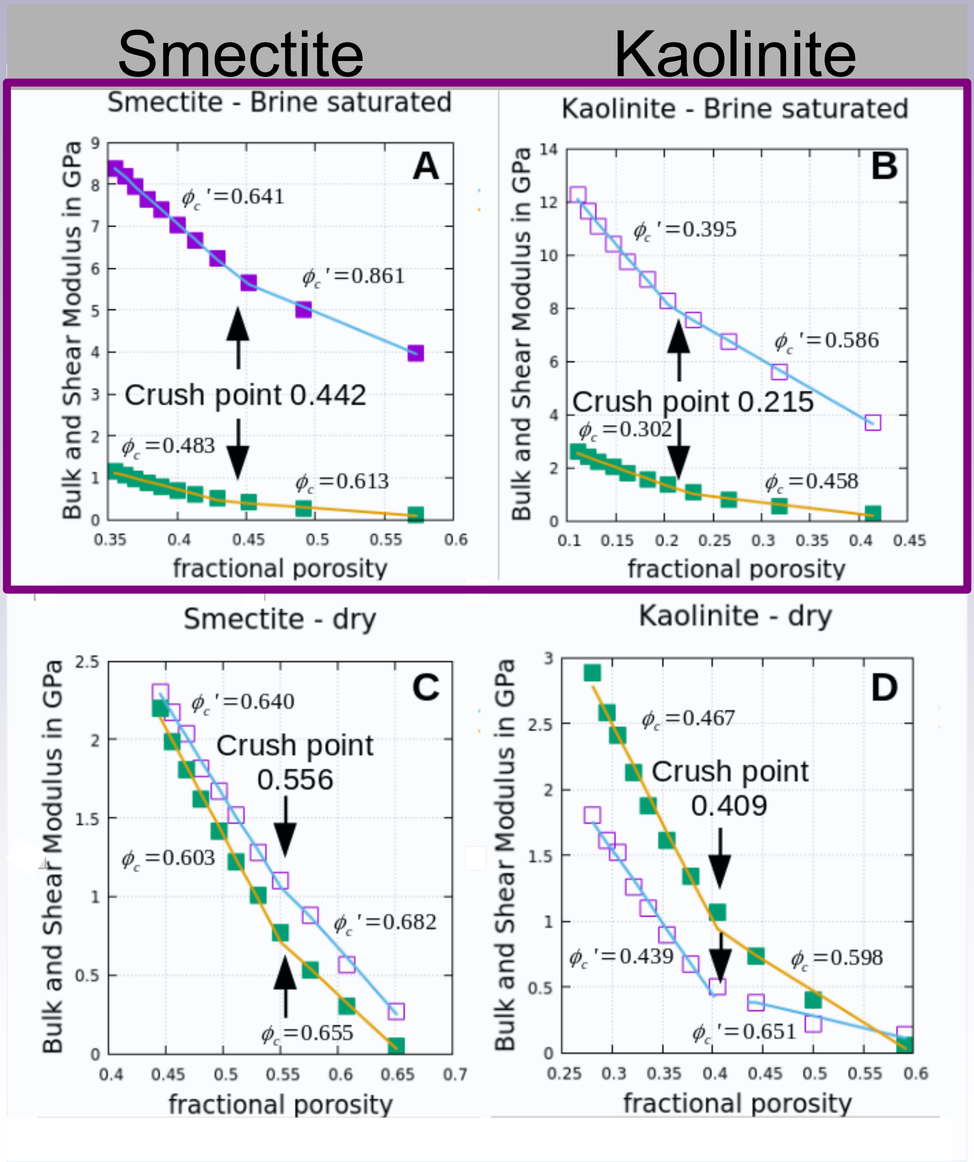
$$s = s(\phi_c, {\phi_c}', K_f, K_m, \rho_f, \rho_m, \mu_m) \rightarrow s = s(\phi_c, {\phi_c}', K_m, \rho_f, \rho_m, \mu_m)$$

Now Focus on the Density Eq.

$$\frac{V_p^2}{a^2} - \frac{V_s^2}{b^2} = 1$$

$$\rho = \frac{C_\rho}{1 - \left(\frac{sV_p}{V_b}\right)^2}$$

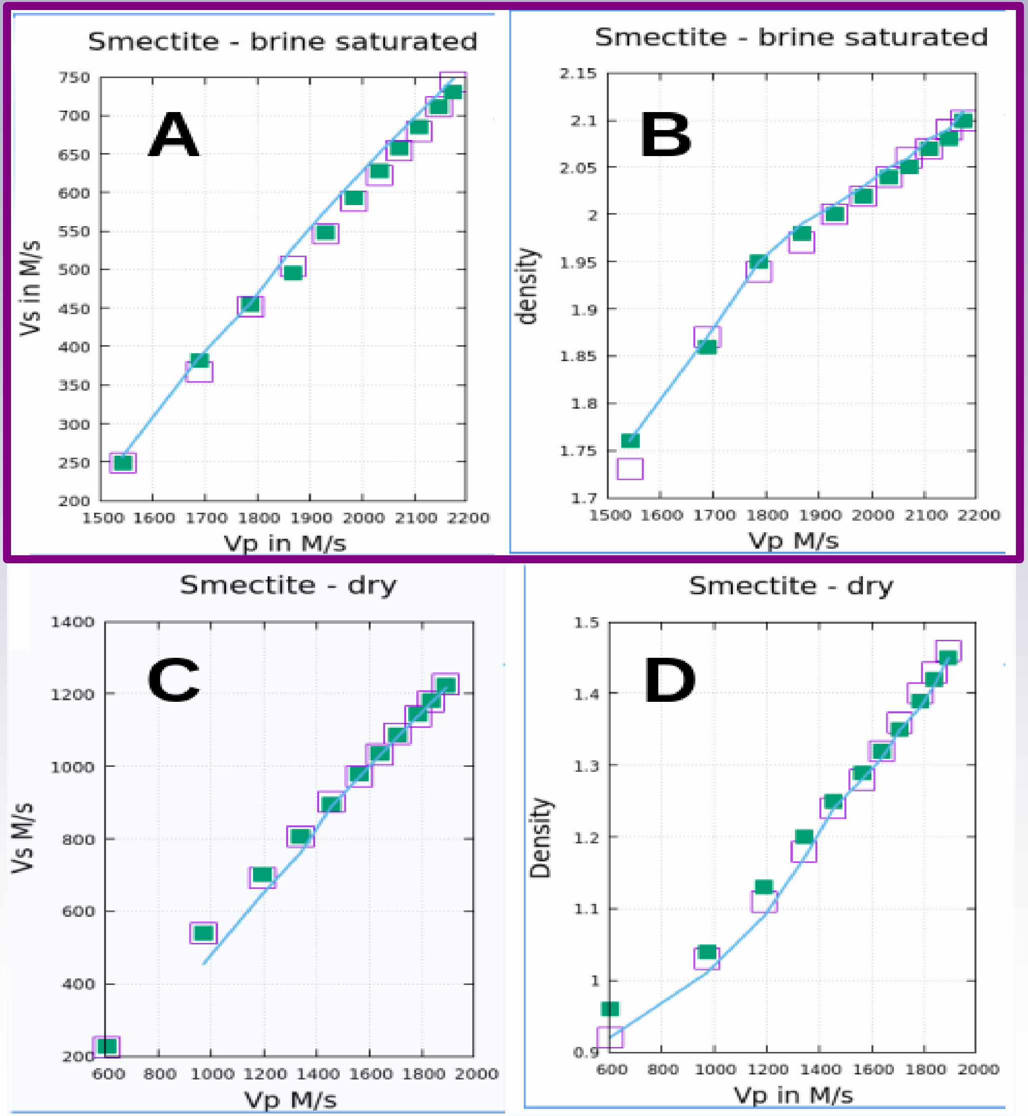
$$V_b = 1500\, m/s$$



Shale
Ultrasonic Freq.

- Mondol's data (Mondol et al., 2008) can be fit quite well to *two connected straight lines.*
- The crush point (grain reorientation) may be the consolidation threshold porosity of Vernik et al. (2010a, 2010b).
- Elastic properties change suddenly at this crush point (consolidation threshold).
- ***Nur's model holds*** on either side of the crush point porosity!
- Bulk modulus slope supplies ϕ_c' (slide 12)
- Shear modulus slope supplies ϕ_c (slide 12)

Evidence SMECTITE



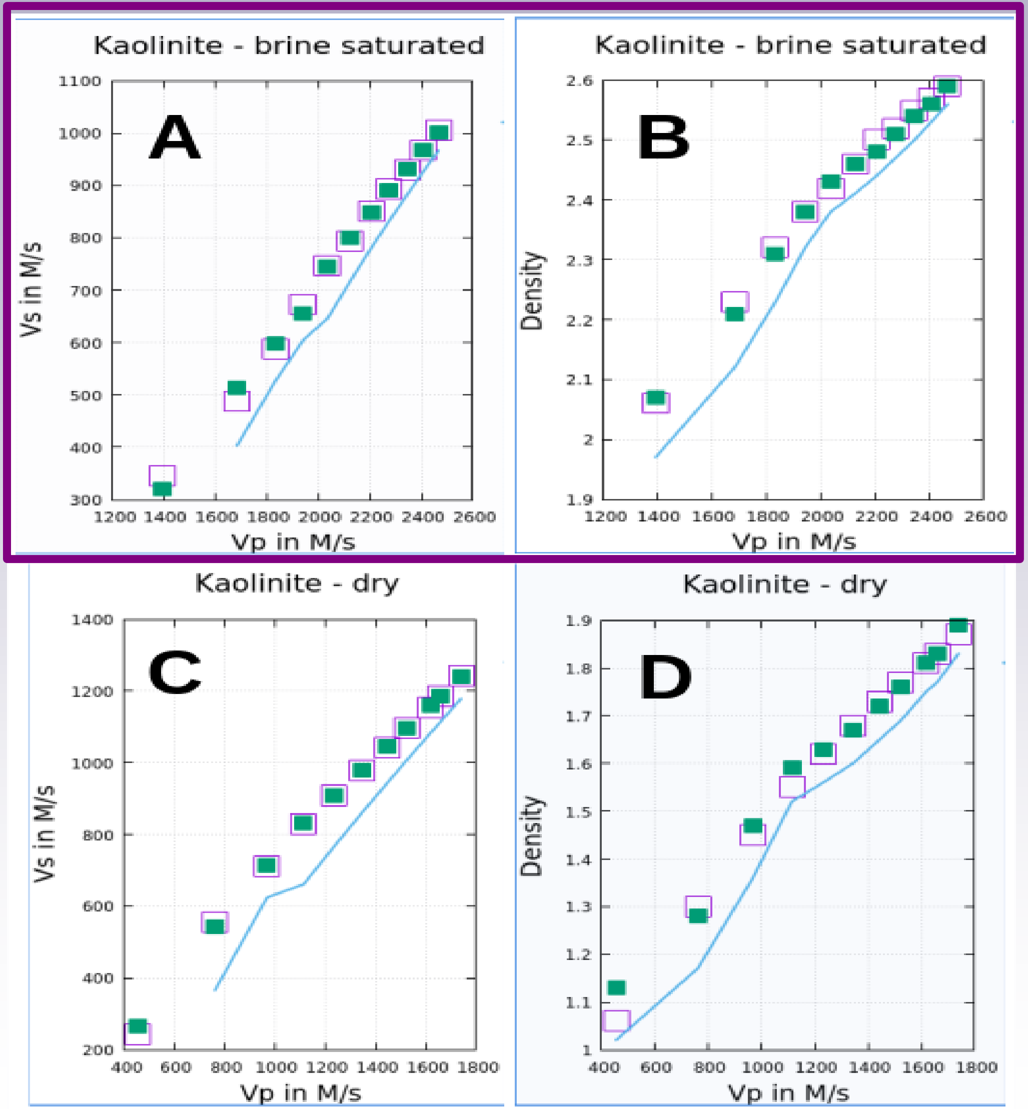
Open squares measured points (Mondol et al., 2008)

Green: Modified Gassman-Nur

Blue: Gassman-Nur using K_f

Wet grain density increased by 2% for 89% pure sample.

Dry grain density decreased by 3% for 89% pure sample.



Evidence Kaolinite

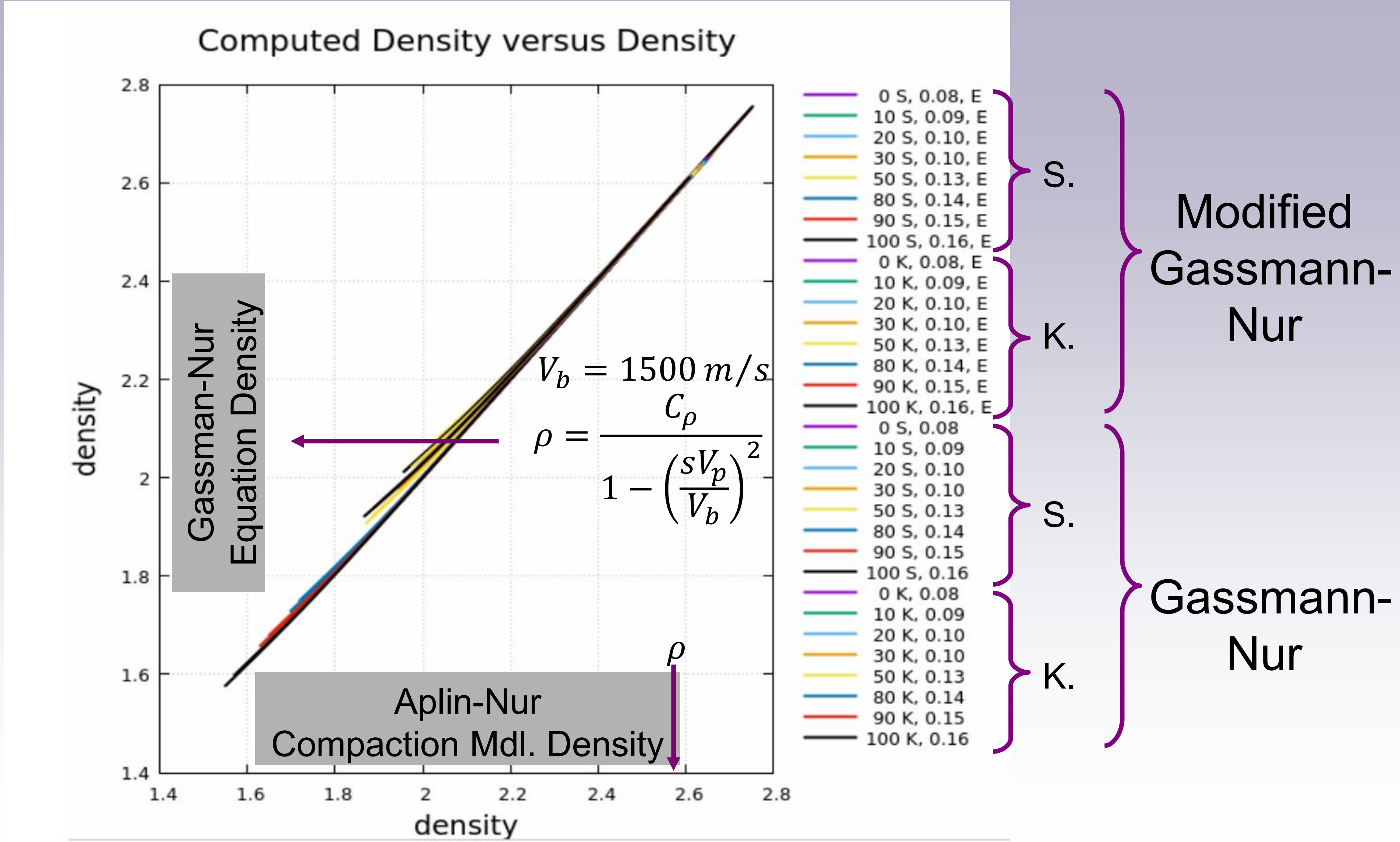
Open squares measured points (Mondol et al., 2008)

Green: Modified Gassman-Nur

Blue: Gassman-Nur using K_f

Wet grain density increased by 6% for 81% pure sample.

Dry grain density decreased by 2% for 81% pure sample.



Computed Density versus Density
density
density
Gassman-Nur
Equation Density
Aplin-Nur
Compaction Mdl. Density
$V_b = 1500\, m/s$
$\rho = \frac{C_\rho}{1 - \left(\frac{sV_p}{V_b}\right)^2}$
ρ
0 S, 0.08, E
10 S, 0.09, E
20 S, 0.10, E
30 S, 0.10, E
50 S, 0.13, E
80 S, 0.14, E
90 S, 0.15, E
100 S, 0.16, E
0 K, 0.08, E
10 K, 0.09, E
20 K, 0.10, E
30 K, 0.10, E
50 K, 0.13, E
80 K, 0.14, E
90 K, 0.15, E
100 K, 0.16, E
0 S, 0.08
10 S, 0.09
20 S, 0.10
30 S, 0.10
50 S, 0.13
80 S, 0.14
90 S, 0.15
100 S, 0.16
0 K, 0.08
10 K, 0.09
20 K, 0.10
30 K, 0.10
50 K, 0.13
80 K, 0.14
90 K, 0.15
100 K, 0.16
S.
K.
S.
K.
Modified
Gassmann-
Nur
Gassmann-
Nur

Conclusions

- The earth compaction model agrees with Gardner’s equation for composite sand shale rocks having high percentage sand content

- The earth compaction model does not agree with Gardner’s equation for composite sand shale rocks having high percentage shale content.

- Evidence is presented that supports the use of equations derived from Gassmann’s equations combined with Nur’s critical porosity model for high frequency low permeability rocks.

- The equation relating density to P-wave velocity derived from combining Gassmann’s equations with Nur’s critical porosity model is shown to agree closely with the compaction models based on combining Nur’s model with Aplin’s equation.

    - *Note: The last two slides provide the formulas used for Bulk Modulus and Density in the compaction models as a function if temperature, salinity, and pressure.*

Bulk Modulus

Bulk modulus of brine in bars as a Fortran function of temperature, 20 < T< 350, in degrees C, salinity, 1 < S < 240,000, in parts per million, and pressure, 98.1 < P <981, in bars, from a fit to Castagna et al. (1993, Fig 21):

```
BlkMod_of_T_S_P(T,S,P)=
: 0.1924E+05 + 0.6646E+01*P -0.6039E-03*P**2 + 0.1650E+03*T
: -0.3031E-01*T*P + 0.1759E-04*T*P**2 -0.1671E+01*T**2
:+ 0.5463E-03*T**2*P -0.2344E-06*T**2*P**2 + 0.4003E-02*T**3
: -0.4519E-06*T**3*P -0.3545E-05*T**4 + 0.4100E-01*S
:+ 0.1776E-04*S*P -0.4226E-08*S*P**2 -0.2771E-03*T*S
: -0.6566E-07*T*S*P + 0.1577E-10*T*S*P**2+0.1476E-05*T**2*S
:+ 0.1796E-09*T**2*S*P -0.2970E-08*T**3*S + 0.1031E-06*S**2
: -0.2194E-10*S**2*P +0.1324E-13*S**2*P**2-0.1425E-09*T*S**2
: -0.7629E-15*T*S**2*P + 0.2335E-12*T**2*S**2 .
```

Density

Density of brine in bars as a Fortran function of temperature, 20 < T < 350, in degrees C, salinity, 1 < S < 300,000, in parts per million, and pressure, 1< P <1000, in bars, from a fit to Castagna et al. (1993, Fig 20):

```
Brine_D_of_T_S_P(T,S,P)= 0.21274E+02*S**(-2)
:+ 0.16674E-01*S**(-2)*P - 0.15228E-04*S**(-2)*P**2
:+ 0.10316E+00*T*S**(-2) - 0.40792E-03*T*S**(-2)*P
:+ 0.28547E-06*T*S**(-2)*P**2 - 0.77854E-03*T**2*S**(-2)
:+ 0.20027E-05*T**2*S**(-2)*P - 0.83005E-09*T**2*S**(-2)*P**2
:+ 0.19105E-05*T**3*S**(-2) - 0.27537E-08*T**3*S**(-2)*P
:+ 0.11955E-13*T**3*S**(-2)*P**2 - 0.21329E+02*S**(-1)
:- 0.16670E-01*S**(-1)*P + 0.15228E-04*S**(-1)*P**2
:- 0.10300E+00*T*S**(-1) + 0.40750E-03*T*S**(-1)*P
:- 0.28515E-06*T*S**(-1)*P**2 + 0.77699E-03*T**2*S**(-1)
:- 0.19976E-05*T**2*S**(-1)*P + 0.82582E-09*T**2*S**(-1)*P**2
:- 0.19067E-05*T**3*S**(-1) + 0.27394E-08*T**3*S**(-1)*P
:+ 0.10624E+01 + 0.43787E-04*P - 0.65442E-08*P**2
:- 0.40447E-03*T - 0.57638E-07*T*P - 0.27188E-11*T*P**2
:- 0.14361E-05*T**2 + 0.86305E-09*T**2*P - 0.98617E-09*T**3 .
:+ 0.52239E-06*S + 0.35447E-10*S*P
:- 0.18153E-13*S*P**2 - 0.12925E-09*T*S - 0.47644E-12*T*S*P
:+ 0.12919E-11*T**2*S
```